

# Polyhedron under Linear Transformations

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## Abstract

The image and the inverse image of a polyhedron under a linear transformation are polyhedrons.

**Keywords:** polyhedron, linear transformation, Sard quotient theorem.

## 1 Introduction

All the linear spaces discussed here are real.

### Definition 1.1.

i.) Suppose that  $X$  is a linear space, a subset  $P$  of  $X$  is said to be a polyhedron if it has the form

$$P = \{x \in X; f_k(x) \leq \lambda_k\},$$

where  $n$  is a positive integer,  $\{f_k\}_{k=1}^n \subset X'$ , and  $\{\lambda_k\}_{k=1}^n \subset \mathbb{R}$ .

If  $\lambda_k = 0$  ( $k = 1, 2, 3, \dots, n$ ), then  $P$  is said to be a polyhedral cone.

ii.) Suppose that  $X$  is a TVS, a subset  $P$  of  $X$  is said to be a closed polyhedron if it has the form

$$P = \{x \in X; f_k(x) \leq \lambda_k\},$$

where  $n$  is a positive integer,  $\{f_k\}_{k=1}^n \subset X^*$ , and  $\{\lambda_k\}_{k=1}^n \subset \mathbb{R}$ .

If  $\lambda_k = 0$  ( $k = 1, 2, 3, \dots, n$ ), then  $P$  is said to be a closed polyhedral cone.

It is obvious that both  $\emptyset$  and  $X$  itself are (closed) polyhedral cones.

## 2 Main Results

Our main results are as follows.

**Theorem 2.1.** Suppose that  $X$  and  $Y$  are linear spaces, and  $T : X \rightarrow Y$  is a linear operator.

i.) If  $A \subset X$  is a polyhedron (polyhedral cone) and  $T$  is **surjective**, then  $T(A)$  is a

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polyhedron (polyhedral cone).

ii.) If  $B \subset Y$  is a polyhedron (polyhedral cone), then  $T^{-1}(B)$  is a polyhedron (polyhedral cone).

**Theorem 2.2.** Suppose that  $X$  and  $Y$  are Fréchet spaces, and  $T : X \rightarrow Y$  is a bounded linear operator.

i.) If  $A \subset X$  is a closed polyhedron (closed polyhedral cone) and  $T$  is **surjective**, then  $T(A)$  is a closed polyhedron (closed polyhedral cone).

ii.) If  $B \subset Y$  is a closed polyhedron (closed polyhedral cone), then  $T^{-1}(B)$  is a polyhedron (closed polyhedral cone).

The conclusions above will be verified in section 4.

### 3 A Lemma

The following conclusion is significant in our proof.

**Lemma 3.1** (Sard Quotient Theorem).

i.) Suppose that  $X, Y$  and  $Z$  are linear spaces, and  $S : X \rightarrow Y, T : X \rightarrow Z$  are linear operators with  $S$  surjective. If  $\ker S \subset \ker T$ , then there exists a uniquely specified linear operator  $R : Y \rightarrow Z$ , such that  $T = RS$ .

ii.) Suppose that  $X, Y$  and  $Z$  are TVS', and  $S : X \rightarrow Y, T : X \rightarrow Z$  are bounded linear operators with  $S$  surjective. If  $X$  and  $Y$  are Fréchet spaces and  $\ker S \subset \ker T$ , then there exists a uniquely specified bounded linear operator  $R : Y \rightarrow Z$ , such that  $T = RS$ .

*Proof.* We will prove only ii.).

Define

$$\begin{aligned}\tilde{S} : X / \ker S &\rightarrow Y \\ [x] &\mapsto Sx,\end{aligned}$$

and

$$\begin{aligned}\tilde{T} : X / \ker S &\rightarrow Z \\ [x] &\mapsto Tx.\end{aligned}$$

Then both  $\tilde{S}$  and  $\tilde{T}$  are well defined (note that  $\ker S \subset \ker T$ ) and bounded. Besides,  $\tilde{S}$  is bijective and  $\tilde{S}^{-1}$  is bounded, since  $X / \ker S$  and  $Y$  are both Fréchet spaces. Now define

$$R = \tilde{T}\tilde{S}^{-1},$$

then it is easy to show that  $R$  satisfies the requirements.

The uniqueness of  $R$  is trivial. ■

## 4 Proofs of Main Results

We will prove only theorem 2.2, because the proof of theorem 2.1 is similar. Only the *polyhedron* case will be discussed.

*Proof of Theorem 2.2.*

i.) Suppose that

$$A = \bigcap_{k=1}^n \{x \in X; f_k(x) \leq \lambda_k\},$$

where  $n$  is a positive integer,  $\{f_k\}_{k=1}^n \subset X^*$ , and  $\{\lambda_k\}_{k=1}^n \subset \mathbb{R}$ . The proof will be presented in four steps.

STEP 1. We will prove that the conclusion holds if

$$\ker T \subset \bigcap_{k=1}^n \ker f_k.$$

In this case, for any  $k \in \{1, 2, 3, \dots, n\}$ , we can choose a functional  $g_k \in Y^*$  such that  $f_k = g_k T$  (Sard quotient theorem). It can be shown without difficulty that

$$T(A) = \bigcap_{k=1}^n \{y \in Y; g_k(y) \leq \lambda_k\}.$$

STEP 2. We will prove that the conclusion holds if  $\dim(\ker T) = 1$ . This is the most critical part of the proof.

Suppose that  $\xi$  is a point in  $\ker T \setminus \{0\}$ . Let

$$\begin{aligned} K_+ &= \{k; 1 \leq k \leq n \text{ and } f_k(\xi) > 0\}, \\ K_- &= \{k; 1 \leq k \leq n \text{ and } f_k(\xi) < 0\}, \\ K_0 &= \{k; 1 \leq k \leq n \text{ and } f_k(\xi) = 0\}. \end{aligned}$$

For any  $i \in K_+$  and  $j \in K_-$ , define

$$h_{ij} = f_i - \frac{f_i(\xi)}{f_j(\xi)} f_j.$$

Then define

$$\begin{aligned} A_1 &= \bigcap_{\substack{i \in K_+, \\ j \in K_-}} \{x \in X; h_{ij}(x) \leq \lambda_i - \frac{f_i(\xi)}{f_j(\xi)} \lambda_j\}, \\ A_2 &= \bigcap_{k \in K_0} \{x \in X; f_k(x) \leq \lambda_k\}. \end{aligned}$$

If  $K_+ = \emptyset$  or  $K_- = \emptyset$ , we take  $A_1$  as  $X$ . Similarly, if  $K_0 = \emptyset$ , we take  $A_2$  as  $X$ . We will prove that  $T(A) = T(A_1 \cap A_2)$ . It suffices to show that  $T(A_1 \cap A_2) \subset T(A)$ .

- If  $K_+ = \emptyset = K_-$ , nothing needs considering.
- If  $K_+ \neq \emptyset = K_-$ , fix a point  $x \in A_1 \cap A_2$ , define

$$s = \min_{i \in K_+} \frac{\lambda_i - f_j(x)}{f_i(\xi)},$$

then it is easy to show that  $x + s\xi \in A$  and  $T(x + s\xi) = Tx$ . The case with  $K_- \neq \emptyset = K_+$  is similar.

- If  $K_+ \neq \emptyset \neq K_-$ , fix a point  $x \in A_1 \cap A_2$ , define

$$t = \max_{j \in K_-} \frac{\lambda_j - f_j(x)}{f_j(\xi)}$$

and consider  $x + t\xi$ . It is obvious that

$$T(x + t\xi) = y$$

and that

$$f_j(x + t\xi) \leq \lambda_j, \quad \forall j \in K_- \cup K_0.$$

Suppose

$$t = \frac{\lambda_{j_0} - f_{j_0}(x)}{f_{j_0}(\xi)} \quad (j_0 \in K_-),$$

then for any  $i \in K_+$ ,

$$\begin{aligned} & f_i(x + t\xi) \\ &= h_{ij_0}(x + t\xi) + \frac{f_i(\xi)}{f_{j_0}(\xi)} f_{j_0}(x + t\xi) \\ &\leq \lambda_i - \frac{f_i(\xi)}{f_{j_0}(\xi)} \lambda_{j_0} + \frac{f_i(\xi)}{f_{j_0}(\xi)} \lambda_{j_0} \\ &= \lambda_i. \end{aligned}$$

Thus  $x + t\xi \in A$ .

It has been shown that  $T(A_1 \cap A_2) \subset T(A)$ , and consequently  $T(A_1 \cap A_2) = T(A)$ . According to STEP 1, the conclusion holds under the assumption  $\dim(\ker T) = 1$ .

STEP 3. We will prove by induction that the conclusion holds if  $\dim(\ker T)$  is finite. If  $\dim(\ker T) = 0$ , then  $T$  is an isomorphism as well as a homeomorphism (inverse mapping theorem), thus nothing needs proving. Now suppose that the conclusion holds when  $\dim(\ker T) \leq n$  ( $n \geq 0$ ). To prove the case with  $\dim(\ker T) = n + 1$ , choose a

point  $\eta$  in  $\ker T \setminus \{0\}$ , find a functional  $F \in X^*$  such that  $F(\eta) = 1$  (Hahn-Banach theorem), and define

$$\begin{aligned}\hat{T} : X &\rightarrow Y \times \mathbb{R} \\ x &\mapsto (Tx, F(x)), \\ \pi : Y \times \mathbb{R} &\rightarrow Y \\ (y, \lambda) &\mapsto y.\end{aligned}$$

Then we have

- $T = \pi\hat{T}$ ;
- $\dim(\ker \hat{T}) = n$ ;
- $\dim(\ker \pi) = 1$ ;
- both  $\hat{T}$  and  $\pi$  are surjective bounded linear operators.

Thus by the induction hypothesis and the conclusion of STEP 2,  $T(A)$  is a closed polyhedron.

STEP 4. Now consider the general case.

Let

$$M = \left( \bigcap_{k=1}^n \ker f_k \right) \bigcap (\ker T),$$

then  $M$  is a closed linear subspace of  $X$ , and therefore  $X/M$  is a Fréchet space. Define

$$\begin{aligned}\tilde{T} : X/M &\rightarrow Y \\ [x] &\mapsto Tx, \\ \tilde{f}_k : X/M &\rightarrow \mathbb{R} \\ [x] &\mapsto f_k(x)\end{aligned}$$

where  $k = 1, 2, 3, \dots, n$ . Then  $\tilde{T}$  and  $\tilde{f}_k$  are well defined,  $T$  is a bounded linear operator from  $X/M$  onto  $Y$ , and  $\{\tilde{f}_k\}_{k=1}^n \subset (X/M)^*$ . Besides, we have

$$\left( \bigcap_{k=1}^n \ker \tilde{f}_k \right) \bigcap (\ker \tilde{T}) = \{0\},$$

which implies that

$$\dim(\ker \tilde{T}) \leq n.$$

Now let

$$\tilde{A} = \bigcap_{k=1}^n \{[x] \in X/M; \tilde{f}_k([x]) \leq \lambda_k\},$$

then

$$T(A) = \tilde{T}(\tilde{A}).$$

From what has been proved, it is easy to show that  $T(A)$  is a closed polyhedron.

Proof of part i.) has been completed.

ii.) This part is much easier. Suppose that

$$B = \bigcap_{k=1}^m \{y \in Y; g_k(y) \leq \mu_k\},$$

where  $m$  is a positive integer,  $\{g_k\}_{k=1}^m \subset Y^*$ , and  $\{\mu_k\}_{k=1}^m \subset \mathbb{R}$ . One can show without difficulty that

$$T^{-1}(B) = \bigcap_{k=1}^m \{x \in X; g_k(Tx) \leq \mu_k\},$$

which is a closed polyhedron in  $X$ . ■

## 5 Remarks

For part i) of theorem 2.2, the completeness conditions are essential. This can be seen from the following examples.

**Example 5.1.** Suppose that  $(Y, \|\cdot\|_Y)$  is an infinite dimensional Banach space, and  $f$  is an unbounded linear functional on it<sup>1</sup>. Let  $X$  has the same elements and linear structure as  $Y$ , but the norm on  $X$  is defined by

$$\|x\|_X = \|x\|_Y + |f(x)|.$$

It is clear that the identify mapping  $I : X \rightarrow Y$  is linear, bounded and bijective. Now consider  $\ker f$ . It is a closed polyhedral cone in  $X$ , while its image under  $I$  is not closed in  $Y$ .

**Example 5.2.** Suppose that  $X$  is  $\ell^1$ . Let  $Y$  has the same elements and linear structure as  $X$ , but the norm on  $Y$  is defined by

$$\|(x_k)\| = \sup_{k \geq 1} |x_k|.$$

Then  $f : (x_k) \mapsto \sum x_k$  is a bounded linear functional on  $X$ , while it is unbounded on  $Y$ . Now consider the identify mapping again.

The preceding examples also imply that inverse mapping theorem and Sard quotient theorem do not hold without completeness conditions.

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<sup>1</sup>For a locally bounded TVS  $Y$ , there exist unbounded linear functionals on  $Y$  provided  $\dim Y = \infty$ . One of them can be constructed as follows: Let  $U$  be a bounded neighborhood of 0, and  $\{e_k; k \geq 1\} \subset U$  be a sequence of linearly independent elements in  $Y$ . Let  $M = \text{Span}\{e_k; k \geq 1\}$ , and define  $g : M \rightarrow \mathbb{R}, \sum \alpha_k e_k \mapsto \sum k \alpha_k$ . Then extend  $g$  to  $Y$ .